dual of Stone representation theorem*

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The Stone representation theorem characterizes a Boolean algebra as a field of sets in a topological space. There is also a dual to this famous theorem that characterizes a Boolean space as a topological space constructed from a Boolean algebra.

Theorem 1. *Let X be a Boolean space. Then there is a Boolean algebra B such that X is homeomorphic to B^*, the dual space of B.*

Proof. The choice for B is clear: it is the set of clopen sets in X which, via the set theoretic operations of intersection, union, and complement, is a Boolean algebra.

Next, define a function $f: X \to B^*$ by

$$f(x) := \{U \in B \mid x \notin U\}.$$

Our ultimate goal is to prove that f is the desired homeomorphism. We break down the proof of this into several stages:

Lemma 1. *f is well-defined.*

Proof. The key is to show that $f(x)$ is a prime ideal in B^* for any $x \in X$. To see this, first note that if $U, V \in f(x)$, then so is $U \cup V \in f(x)$, and if W is any clopen set of X, then $U \cap W \in f(x)$ too. Finally, suppose that $U \cap V \in f(x)$. Then $x \in X - (U \cap V) = (X - U) \cup (X - V)$, which means that $x \notin U$ or $x \notin V$, which is the same as saying that $U \in f(x)$ or $V \in f(x)$. Hence $f(x)$ is a prime ideal, or a maximal ideal, since B is Boolean. □

Lemma 2. *f is injective.*

Proof. Suppose $x \neq y$, we want to show that $f(x) \neq f(y)$. Since X is Hausdorff, there are disjoint open sets U, V such that $x \in U$ and $y \in V$. Since X is also totally disconnected, U and V are unions of clopen sets. Hence we may as well assume that U, V clopen. This then implies that $U \in f(y)$ and $V \in f(x)$. Since $U \neq V$, $f(x) \neq f(y)$. □

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Lemma 3. *f is surjective.*

Proof. Pick any maximal ideal I of B^*. We want to find an $x \in X$ such that $f(x) = I$. If no such x exists, then for every $x \in X$, there is some clopen set $U \in I$ such that $x \in U$. This implies that $\bigcup I = X$. Since X is compact, $X = \bigcup J$ for some finite set $J \subseteq I$. Since I is an ideal, and X is a finite join of elements of I, we see that $X \in I$. But this would mean that $I = B^*$, contradicting the fact that I is a maximal, hence a proper ideal of B^*. $\square$

Lemma 4. *f and f^{-1} are continuous.*

Proof. We use a fact about continuous functions between two Boolean spaces:

> a bijection is a homeomorphism iff it maps clopen sets to clopen sets (proof here).

So suppose that U is clopen in X, we want to prove that $f(U)$ is clopen in B^*. In other words, there is an element $V \in B$ (so that V is clopen in X) such that

$$f(U) = M(V) = \{M \in B^* \mid V \notin M\}.$$

This is because every clopen set in B^* has the form $M(V)$ for some $V \in B^*$ (see the lemma in this entry). Now, $f(U) = \{f(x) \mid x \in U\} = \{f(x) \mid U \notin f(x)\} = \{M \mid U \notin M\}$, the last equality is based on the fact that f is a bijection. Thus by setting $V = U$ completes the proof of the lemma. $\square$

Therefore, f is a homemorphism, and the proof of theorem is complete. $\square$