characteristic polynomial of algebraic number*

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Let ϑ be an algebraic number of degree n, $f(x)$ its minimal polynomial and

$$\vartheta_1 = \vartheta,\ \vartheta_2,\ \ldots,\ \vartheta_n$$

its algebraic conjugates.

Let α be an element of the number field $\mathbb{Q}(\vartheta)$ and

$$r(x)\ :=\ c_0 + c_1x + \ldots + c_{n-1}x^{n-1}$$

the canonical polynomial of α with respect to ϑ. We consider the numbers

$$r(\vartheta_1)\ =\ \alpha\ :=\ \alpha^{(1)},\quad r(\vartheta_2)\ :=\ \alpha^{(2)},\quad \ldots,\quad r(\vartheta_n)\ :=\ \alpha^{(n)} \tag{1}$$

and form the equation

$$g(x)\ :=\ \prod_{i=1}^{n}[x-r(\vartheta_i)]\ =\ (x-\alpha^{(1)})(x-\alpha^{(2)})\cdots(x-\alpha^{(n)})\ =\ x^n+g_1x^{n-1}+\ldots+g_n\ =\ 0,$$

the roots of which are the numbers (1) and only these. The coefficients g_i of the polynomial $g(x)$ are symmetric polynomials in the numbers $\vartheta_1,\ \vartheta_2,\ \ldots,\ \vartheta_n$ and also symmetric polynomials in the numbers $\alpha^{(i)}$. The fundamental theorem of symmetric polynomials implies now that the symmetric polynomials g_i in the roots ϑ_i of the equation $f(x) = 0$ belong to the ring determined by the coefficients of the equation and of the canonical polynomial $r(x)$; thus the numbers g_i are rational (whence the degree of α is at most equal to n).

It is not hard to show (see the entry degree of algebraic number) of that the degree k of α divides n and that the numbers (1) consist of α and its algebraic conjugates $\alpha_2,\ \ldots,\ \alpha_k$, each of which appears in (1) exactly $\frac{n}{k} = m$ times. In fact, $g(x) = [a(x)]^m$ where $a(x)$ is the minimal polynomial of α (consequently,

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the coefficients g_i are integers if α is an algebraic integer).

The polynomial $g(x)$ is the *characteristic polynomial* (in German *Hauptpolynom*) of the element α of the algebraic number field $\mathbb{Q}(\vartheta)$ and the equation $g(x) = 0$ the *characteristic equation* (*Hauptgleichung*) of α. See the independence of characteristic polynomial on primitive element.

So, the roots of the characteristic equation of α are $\alpha^{(1)}$, $\alpha^{(2)}$, …, $\alpha^{(n)}$. They are called the $\mathbb{Q}(\vartheta)$-*conjugates* of α; they all are algebraic conjugates of α.