Proof of Bonferroni Inequalities*

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Definitions and Notation. A measure space is a triple $(X, \Sigma, \mu)$, where $X$ is a set, $\Sigma$ is a σ-algebra over $X$, and $\mu: \Sigma \to [0, \infty]$ is a measure, that is, a non-negative function that is countably additive. If $A \in \Sigma$, the characteristic function of $A$ is the function $\chi_A: X \to \mathbb{R}$ defined by $\chi_A(x) = 1$ if $x \in A$, $\chi_A(x) = 0$ if $x \notin A$. A unimodal sequence is a sequence of real numbers $a_0, a_1, \ldots, a_n$ for which there is an index $k$ such that $a_i \leq a_{i+1}$ for $i < k$ and $a_i \geq a_{i+1}$ for $i \geq k$.

The proof of the following easy lemma is left to the reader:

**Lemma 1.** If $a_0 \leq a_1 \leq \ldots \leq a_k \geq a_{k+1} \geq \ldots \geq a_n$ is a unimodal sequence of non-negative real numbers with $\sum_{i=0}^{n} (-1)^i a_i = 0$, then $\sum_{j=0}^{k} (-1)^j a_i \geq 0$ for even $j$ and $\leq 0$ for odd $j$.

Since the binomial sequence $(\binom{n}{i})_{0 \leq i \leq n}$ with integer $a > 0$ and integer $n \geq a$ satisfies the hypothesis of Lemma ??, we have:

**Corollary 1.** If $a$ is a positive integer, $\sum_{j=0}^{l} (-1)^j \binom{n}{i} \geq 0$ for even $j$ and $\leq 0$ for odd $j$.

**Lemma 2.** Let $(A_i)_{1 \leq i \leq n}$ be a sequence of sets and let $X = \bigcup_{1 \leq i \leq n} A_i$. For $x \in X$, let $I(x)$ be the set of indices $j$ such that $x \in A_j$. If $1 \leq k \leq n$,

$$\sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} \chi_{A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}}(x) = \binom{|I(x)|}{k}$$

for all $x \in X$.

**Proof.** $\chi_{A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}}(x) = 1$ if $\{i_1, i_2, \ldots, i_k\} \subseteq I(x)$, and $= 0$ otherwise. Therefore the sum equals the number of $k$-subsets of $I(x)$, which is $\binom{|I(x)|}{k}$. \[\square\]

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Theorem 1. Let \((X, \Sigma, \mu)\) be a measure space. If \((A_i)_{1 \leq i \leq n}\) is a finite sequence of measurable sets all having finite measure, and

\[
S_j = \mu(A_1 \cup A_2 \cup \ldots \cup A_n) + \sum_{k=1}^{j} (-1)^k \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} \mu(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k})
\]

then \(S_j \geq 0\) for even \(j\), and \(\leq 0\) for odd \(j\). Moreover, \(S_n = 0\) (Principle of Inclusion-Exclusion).

Proof. Let \(Y = \bigcup_{1 \leq i \leq n} A_i\).

\[
S_j = \int_Y d\mu + \sum_{k=1}^{j} (-1)^k \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} \int_Y \chi_{A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}} d\mu
\]

\[
= \int_Y d\mu + \sum_{k=1}^{j} (-1)^k \int_Y \left( \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} \chi_{A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}} \right) d\mu
\]

By Lemma ??,

\[
S_j = \int_Y d\mu + \sum_{k=0}^{j} (-1)^k \int_Y \binom{|I(x)|}{k} d\mu
\]

\[
= \sum_{k=0}^{j} (-1)^k \int_Y \binom{|I(x)|}{k} d\mu
\]

\[
= \int_Y \sum_{k=0}^{j} (-1)^k \binom{|I(x)|}{k} d\mu
\]

Since \(|I(x)| > 0\) for \(x \in Y\), it follows from Corollary ?? that, in the last integral, the integrand is \(\geq 0\) for even \(j\) and \(\leq 0\) for odd \(j\). Therefore the same is true for the integral itself. In addition, the integrand is identically 0 for \(j = n\), hence \(S_n = 0\). 

This proof shows that at the heart of Bonferroni’s inequalities lie similar inequalities governing the binomial coefficients.