linear space*

CWoo[†]

2013-03-22 3:33:50

A *linear space* is a near-linear space $\mathscr{S} = (\mathcal{P}, \mathcal{L})$ in which every pair of distinct points are on exactly one line.

Note: The usage of the term has little relation to its occasional appearance in linear algebra as a synonym for a vector space.

Note: The convention for drawing finite linear spaces is to ignore two-point lines. Thus, a linear space with three points and three lines is drawn

rather than

Examples:

1. We have seen in the parent entry that a graph can be thought of as a near-linear space in which every line contains two points. A complete graph is then a linear space.

2. Let $\mathbb{F}$ be a finite field. Let $\mathcal{P}$ be the elements in the Cartesian product $\mathbb{F} \times \mathbb{F}$. The solutions to a linear equation

$$\{(x, y) \in \mathcal{P} \mid ax + by = c\}$$

for some $a, b, c \in \mathbb{F}$, where a and b are not both zero, form a line in $\mathcal{L}$. Since any two points determine a unique line, $\mathscr{A} = (\mathcal{P}, \mathcal{L})$ is a linear space, called the affine plane over $\mathbb{F}$.

3. Any projective plane is a linear space. Conversely, any linear space in which every two lines meet, and there exists a quadrangle is a projective plane.

4. A *near-pencil* is a finite linear space with the following diagram:

*⟨*LinearSpace*⟩ created: ⟨*2013-03-2*⟩ by: ⟨*CWoo*⟩ version: ⟨*42174*⟩ Privacy setting: ⟨*1*⟩ ⟨*Definition*⟩ ⟨*51A45*⟩ ⟨*51A05*⟩ ⟨*05C65*⟩

P

P_1 P_2 $\cdots$ P_{n-1} P_n ℓ

In other words, a near-pencil consists of $n+1$ points and $n+1$ lines, where $n \geq 2$, such that n points lie on one line ℓ, and the remaining point P lies on n 2-point lines. Of course, on any 2-point line, the point other than P must be on ℓ.

A fundamental fact concerning finite linear space is the De Bruijn Erdős theorem, which states that given a finite linear space $\mathscr{S} = (\mathcal{P}, \mathcal{L})$ with at least two lines, then $|\mathcal{P}| \leq |\mathcal{L}|$, and if the equality occurs, then $\mathscr{S}$ is either a projective plane or a near-pencil.

5. Let $k_1, \ldots, k_n$ be positive integers greater than 2. A $(k_1, \ldots, k_n)$-*star* is a linear space such that there are n lines of sizes $k_1, \ldots, k_n$ respectively, which are concurrent, and all other lines are 2-point lines. By convention, we order the integers so that $k_1 \leq \cdots \leq k_n$. When $n = 2$, we call the space a (k_1, k_2)-cross. Below are diagrams of a $(3, 3, 3, 3)$-star and a $(4, 5)$-cross:

6. There are five non-isomorphic five-point linear spaces:

Corresponding to the diagrams above, the spaces have respectively 1, 5, 6, 8, and 10 lines. Of these, the second is a near-pencil, the third is a $(3, 3)$-cross, and the last is a complete graph.

Some properties:

1. In a linear space $\mathscr{S} = (\mathcal{P}, \mathcal{L})$,

$$\sum_{\ell \in \mathcal{L}} \binom{|\ell|}{2} = \binom{|\mathcal{P}|}{2}.$$

2. Let p be an arbitrary point in a linear space,

$$\sum_{\ell \ni p} (|\ell| - 1) = |\mathcal{P}| - 1$$

   where the sum is taken over all lines containing p. This holds because given any point, this point forms exactly one line with every other point, so $|\ell| - 1$ counts the number of points p shares in line ℓ. Summing over all lines gives all the points except p.

References

[1] L. M. Batten, A. Beutelspacher *The Theory of Finite Linear Spaces*, Cambridge University Press (2009)