evolute of cycloid*

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2013-03-22 3:39:47

We shall determine the evolute of the cycloid

$$x \;=\; a(u-\sin u), \qquad y \;=\; a(1-\cos u), \tag{1}$$

where the parametre u is the rolling angle of the circle with radius a forming the cycloid.

The parametric equations of the evolute of a curve $x = x(u),\; y = y(u)$ are

$$\xi \;=\; x-\varrho\sin\alpha, \qquad \eta \;=\; y+\varrho\cos\alpha \tag{2}$$

with α the slope angle of the tangent and ϱ the radius of curvature of the given curve in the point $(x,\,y)$:

$$\varrho \;=\; \frac{(x'^{\,2}+y'^{\,2})^{3/2}}{x'y''-x''y'}$$

In the case of the cycloid (1), we have

$$x' \;=\; a(1-\cos u), \qquad y' \;=\; a\sin u, \qquad x'' \;=\; a\sin u, \qquad y'' \;=\; a\cos u.$$

Now we get

$$x'^{\,2}+y'^{\,2} \;=\; 2a^2(1-\cos u) \;=\; 4a^2\sin^2\frac{u}{2},$$
$$x'y''-x''y' \;=\; a^2(\cos u-1) \;=\; -2a^2\sin^2\frac{u}{2},$$

and thus the radius of curvature (red in the diagram) is

$$\varrho \;=\; -4a\sin\frac{u}{2}. \tag{3}$$

We utilised the identity

$$1-\cos u \;=\; 2\sin^2\frac{u}{2} \tag{4}$$

*⟨*EvoluteOfCycloid*⟩ created: ⟨*2013-03-2*⟩ by: ⟨*pahio*⟩ version: ⟨*42237*⟩ Privacy setting: ⟨*1*⟩ ⟨*Derivation*⟩ ⟨*53A04*⟩

(see the half angle formula of sine in the goniometric formulas). It is easy to show that the point, where the circle touches the x-axis, bisects the radius of curvature (which lies on the normal line at the point $(x,\, y)$ of the cycloid).

Using then the derivative for parametric form, we obtain

$$\tan\alpha \;=\; \frac{dy}{dx} \;=\; \frac{y'}{x'} \;=\; \frac{\sin u}{1-\cos u}.$$

which implies

$$\sin\alpha \;=\; \cos\frac{u}{2}, \quad \cos\alpha \;=\; \sin\frac{u}{2}, \tag{5}$$

Substituting all needed expressions (1), (3), (5) into (2) and simplifying, we arrive at the result

$$\xi \;=\; a(u+\sin u), \qquad \eta \;=\; -a(1-\cos u). \tag{6}$$

The equations (6) represent **the evolute of the given cycloid. But it is also a cycloid**, congruent to the original one, which has been translated the distance πa to the left and the distance $2a$ downwards; this is seen when one performs in (6) the substitution $u = v - \pi$; then (6) reads

$$\xi \;=\; a(v+\sin v) - \pi a, \qquad \eta \;=\; a(1-\cos v) - 2a.$$